Tail asymptotics for a Lévy-driven tandem queue with an intermediate input and its extensions

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Our plan

- 2 node tandem fluid queue
- What is our interest ?
- Modeling assumptions and basic facts
- Our approach
- The convergence domain of φ
- Exact asymptotics by complex inversions
- Weak asymptotics for the Lévy input case
- Remarks on extensions
- References

2 node tandem fluid queue

$X_2(t)$ is called an intermediate input.

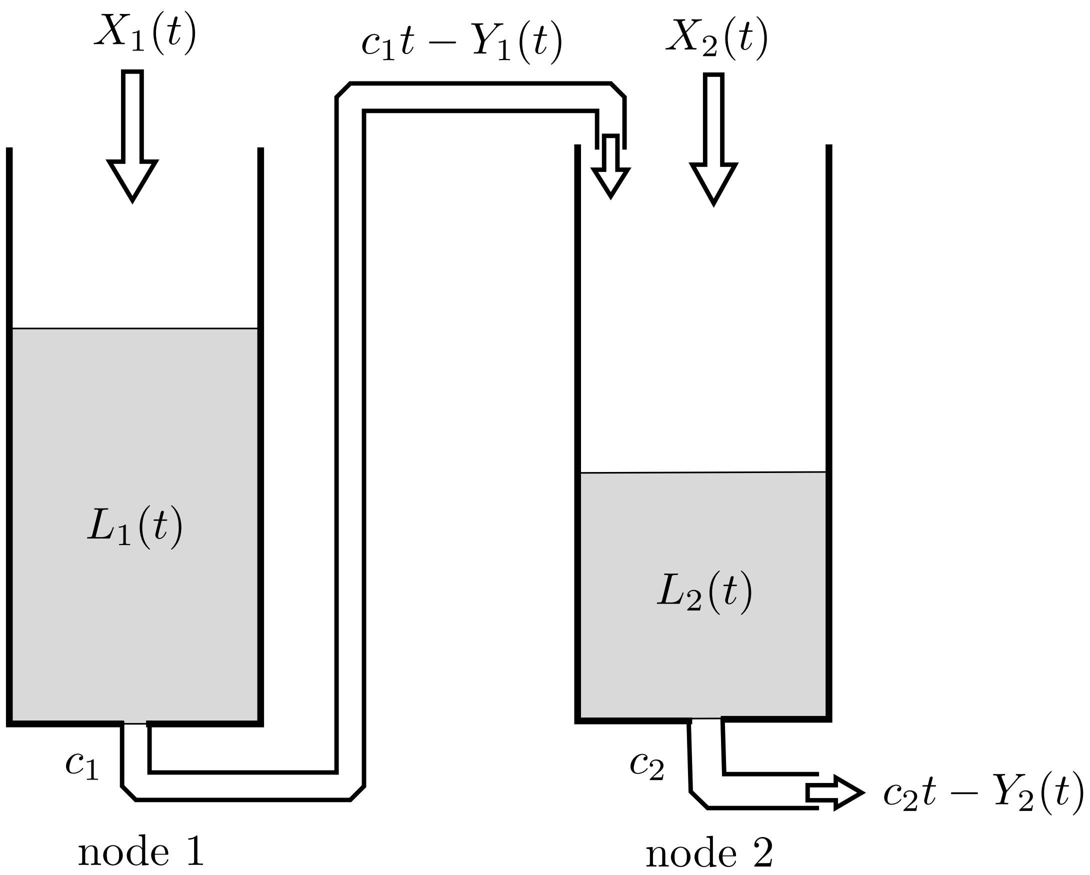


Figure 1: The buffer contents $L_i(t)$, accumulated inputs $X_i(t)$ and outputs $c_i t - Y_i(t)$ at time t for $i = 1, 2$.

What is our interest ?

For the two node tandem fluid queue, we assume:

- The second node also has an exogenous input.
- The exogenous inputs are independent and subject to the **spectrally positive Lévy process**.
- Each node has a constant release rate.

We like to see how inputs influence its stationary distributions.

- No closed form result is available*.
- Decay rates are known for the **Brownian inputs**.

$\Rightarrow$ Our interest is in **exact tail asymptotics** of the stationary distribution, particularly of the second buffer content.

$\ast$ This is different from the pure tandem queue.

Background of this talk

This study is stimulated by

- the recent work of Lieshout and Mandjes (2008), which considers the pure tandem fluid queue.
- exact asymptotics on the double QBD of Miyazawa (2009)
- exact asymptotics on the random walk on a positive quadrant of Foley and McDonald (2005)

Those papers apply techniques which are **different from the large deviations**, particularly, sample path large deviations.

We are interested in the above approaches because

- exact asymptotics can be considered,
- they might be applicable for a higher dimensional case.

No explicit result is known for more than 2-node networks.

About the Lévy process

A real valued process $\{X(t)\}$ is said to be 1-dimensional Lévy process if it has independent and stationary increments, which can be decomposed into the following three components.

- $at + \sigma B(t)$: Brownian motion with drift a and variance σ^2.
- $J^{(0)}(t) = \int_0^t \int_0^1 x(\Lambda(du, dx) - du\nu(dx))$: Martingale with jumps not greater than 1.
- $J^{(1)}(t) = \int_0^t \int_{1+}^{\infty} x\Lambda(du, dx)$: Compound Poisson process with jumps greater than 1,

where random measure Λ is a pure jump component of $\{X(t)\}$, and $du\nu(dx) = E(\Lambda(du, dx))$. ν is called a spectral measure.

We assume that $X(t)$ only has **positive jumps**. This may be reasonable for input processes.

Lévy exponent

The moment generating function of Lévy process $X(t)$ has the following form.

$$E(e^{\theta X(t)}) = e^{t\kappa(\theta)}, \qquad \Re\theta \leq 0.$$

where $\kappa(\cdot)$ is called an Lévy exponent, and given by

$$\kappa(\theta) = a\theta + \frac{1}{2}\sigma^2\theta^2 + \kappa^{(0)}(\theta) + \kappa^{(1)}(\theta). \tag{1}$$

where

$$\kappa^{(0)}(\theta) = \int_0^1 (e^{\theta x} - 1 - \theta x)\nu(dx),$$

$$\kappa^{(1)}(\theta) = \int_{1+}^{\infty} (e^{\theta x} - 1)\nu(dx).$$

We put subscript i for characteristics for node i. For example, $X_i(t), a_i, \sigma_i, \kappa_i(\theta), \kappa_i^{(0)}(\theta), \kappa_i^{(1)}(\theta)$.

Sample path of the tandem fluid queue

For node i, let $X_i(t)$ be the accumulated input at time $t \geq 0$, $L_i(t)$ be the buffer content and c_i be a release rate, then

$$L_1(t) = L_1(0) + X_1(t) - c_1 t + Y_1(t), \tag{2}$$

$$L_2(t) = L_2(0) + X_2(t) + c_1 t - Y_1(t) - c_2 t + Y_2(t), \tag{3}$$

where, $Y_i(t)$ is a nondecreasing process which regulates $L_i(t)$ to be nonnegative, and called a **regulator**. Then, $(L_1(t), L_2(t))$ is called a reflecting process generated by $(X_1(t) - c_1 t, X_2(t) + c_1 t - c_2 t)$ through reflection matrix

$$R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Since $X_i(t)$ has no negative jump, $Y_i(t)$ **is continuous in** t.

Stationary equations

Let $\lambda_i \equiv E(X_i(1))$, and assume the stability condition:

$$\lambda_1 < c_1, \qquad \lambda_1 + \lambda_2 < c_2,$$

then the stationary distribution exists. Denote it by ν. Let $\boldsymbol{L} \equiv (L_1, L_2)$ be a random vector subject to ν, and define

$$\varphi(\theta_1, \theta_2) = E(e^{\theta_1 L_1 + \theta_2 L_2}), \quad \varphi_i(\theta_j) = E_\nu\left(\int_0^1 e^{\theta_j L_j(t)} Y_i(dt)\right),$$

where E_ν stands for the conditional expectation given $\boldsymbol{L}(0)$ having ν. Applying **Itô's formula*** to (2) and (3), we have

Proposition 1 For $\theta_1, \theta_2 \leq 0$,

$$\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2) = (\theta_1 - \theta_2)\varphi_1(\theta_2) + \theta_2\varphi_2(\theta_1), \tag{4}$$

where, $\gamma(\theta_1, \theta_2) = c_1\theta_1 + (c_2 - c_1)\theta_2 - \kappa_1(\theta_1) - \kappa_2(\theta_2)$.

$*$ We can also use **Kella-Whitt martingale** for this derivation.

Rough and exact asymptotics

Since stationary distribution ν is two dimensional, there may be many ways to consider its asymptotics. We consider the tails of $L_1, L_2, d_1 L_1 + d_2 L_2$, where d_i's are positive constants.

We define their asymptotics in the following way. For positive function $g(x)$ for $x \in [0, \infty)$,

$$\overline{\alpha} = -\limsup_{x\to\infty} \frac{1}{x} \log g(x), \qquad \underline{\alpha} = -\liminf_{x\to\infty} \frac{1}{x} \log g(x)$$

are called **upper and lower decay rates** of $g(x)$. If $\alpha = \overline{\alpha} = \underline{\alpha}$, then α is called a **decay rate**. If there is a positive function h such that

$$\lim_{x\to\infty} \frac{g(x)}{h(x)} = 1,$$

then $h(x)$ is said to be **exact asymptotics** of $g(x)$, and write it as $g(x) \sim h(x)$.

What is known around this model

- Brownian networks (BN) and Lévy networks (LN)

  Reflection map for BN – Harrison and Reiman (1981)

  LST for tandem LN – Kella and Whitt (1992)

  LST for out-tree LN – Dębicki, Dieker and Rolski (2007)

  Sample path LD – Majewski (1998)

  Rough asymptotics for 2 node BN

  – Avram, Dai and Hasenbein (2001)

- Discrete-time queueing networks

  Rough asymptotics for intree networks – Chang (1995)

  Rough and Exact asymptotics for 2 node networks

  – Borovkov and Mogul'skii (2001)

Already mentioned about Lieshout and Mandjes (2008), Foley and McDonald (2005) and Miyazawa (2009).

Our approach

Tail asymptotics of the stationary distributions of queueing networks have been studied by

- Sample path large deviations (**[1, 3]**)
- Spectra or convergence radius (**[5]**).
- Markov renewal theorem.
- Complex analysis and/or local limit theorems.
- Tauberial theorems when LST is known (**[4]**).

Our approach of **[6]** is close to the last one, but

1. we first find singular points through the **convergence domain** of the joint moment generating function φ,
2. then apply **complex inversions** for asymptotics due to **[2]**.

Stationary distribution of node 1

Let $\theta_2 = 0$ in (4), then

$$(c_1\theta_1 - \kappa_1(\theta_1))\varphi(\theta_1, 0) = \theta_1\varphi_1(0).$$

Thus, we have **Pollaczek-Khinchine formula**:

$$\varphi(\theta_1, 0) = \frac{(c_1 - \lambda_1)\theta_1}{(c_1\theta_1 - \kappa_1(\theta_1))}. \tag{5}$$

From this formula, if there exists $\alpha_1 > 0$ satisfying

$$c_1\alpha_1 = \kappa_1(\alpha_1),$$

then α_1 is the rough decay rate of $P(L_1 > x)$. Furthermore, if, for some $\epsilon > 0$, $k_1(\theta)$ is finite for $\theta < \alpha_1 + \epsilon$, then

$$P(L_1 > x) \sim Ce^{-\ \ {}_1 x}$$

for some constant $C > 0$ since $z = \alpha_1$ is a simple pole of the analytic function $\varphi(z, 0)$.

Brownian input case: the domain of φ

For simplicity, assume that $X_i(t)$ is continuous in time t, that is, a Brownian motion. Let $r_1 = c_1 - \lambda_1, \;\; r_2 = c_2 - c_1 - \lambda_2$, then

$$\gamma(\theta_1, \theta_2) = r_1\theta_1 + r_2\theta_2 - \frac{1}{2}(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2),$$

and therefore stationary equation (4) becomes

$$\begin{aligned}(r_1 z_1 + r_2 z_2 - \frac{1}{2}\sigma_1^2(z_1^2 + \sigma_2^2 z_2^2))\varphi(z_1, z_2) \\ = (z_1 - z_2)\varphi_1(z_2) + z_2\varphi_2(z_1). \qquad (6)\end{aligned}$$

Thus, we have the following facts.

Lemma 1 The left side of (6) is **an analytic function of two variables** $\Longleftrightarrow$ $\varphi_1(z)$ and $\varphi_2(z)$ are **analytic functions of** z.

Remark: $f(z_1, z_2)$ is said to be an analytic function of two variable if $f(z_1, z_2)$ is analytic for each z_i.

Further auxiliary lemmas

From the stationary equation (6),

$$\varphi_1(\theta_2) = (\frac{1}{2}\sigma_2^2\theta_2 - r_2)\varphi(0,\theta_2) + r_1 + r_2.$$

Since

$$\varphi(0,\theta_2) = \frac{\varphi_1(\theta_2) - \varphi_1(z_2)}{\frac{1}{2}\sigma_2^2\theta_2 - r_2},$$

$z_2 = \frac{2r_2}{\sigma_2^2}$ is a removable singular point of $\varphi(0,z)$. Hence,

Lemma 2 $\varphi_1(z)$ and $\varphi(0,z)$ have the same singularity.

Lemma 3 $\varphi_2(\theta)$ is finite for $\theta < \alpha_1$, where $\alpha_1 = \frac{2r_1}{\sigma_1^2}$.

Remark 1 The domain of $\varphi_2(\theta)$ may be larger than $\theta < \alpha_1$.

Ideas for finding the domain

Let $\mathcal{D} = \{\boldsymbol{\theta} \in I\!\!R^2; \varphi(\theta_1, \theta_2) < \infty\}$, which is the convergence domain of φ. How we can get it from stationary equation (6).

- **Boundary free kernel** : Let $\Gamma_0 = \{\boldsymbol{\theta} \in I\!\!R^2; \gamma(\boldsymbol{\theta}) = 0\}$. Since $\gamma(\boldsymbol{\theta}) = r_1\theta_1 + r_2\theta_2 - \frac{1}{2}(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2)$, Γ_0 is ellipse. Denote its inside by Γ_+ and outside by Γ_-.
- **Signs** : Since $\gamma(\boldsymbol{\theta})\varphi(\boldsymbol{\theta}) + (\theta_2 - \theta_1)\varphi_1(\theta_2) = \theta_2\varphi_2(\theta_1)$, if $\boldsymbol{\theta} \in \Gamma_+$ and $\theta_2 > \theta_1$, then $\varphi_2(\theta_1)$ is finite if and only if $\varphi(\boldsymbol{\theta})$ and $\varphi_1(\theta_2)$ are finite, that is, $\boldsymbol{\theta} \in \mathcal{D}$. Furthermore, $\theta_1 < \alpha_1 \Leftrightarrow \varphi(\theta_1, 0) < \infty \Rightarrow \varphi_2(\theta_1) < \infty$. Hence, $\boldsymbol{\theta} \in \Gamma_+, \theta_2 > \theta_1, \theta_1 < \alpha_1 \Rightarrow \boldsymbol{\theta} \in \mathcal{D}$.
- **Check all combinations** : Γ_0^+ or Γ_0^- and the signs of $\theta_2 - \theta_1$ and θ_1, $2^3 = 8$ cases in total are examined.

Notation for the convergence domain

Since $\gamma(\theta_1, \theta_2) = 0$ is ellipse, we can define:

$$(\theta_1^{\max}, \theta_2^{\max}) = \arg \max_{(\theta_1,\theta_2)} \{\theta_2; \gamma(\theta_1, \theta_2) = 0\},$$

$$(\theta_1^{\min}, \theta_2^{\min}) = \arg \min_{(\theta_1,\theta_2)} \{\theta_2; \gamma(\theta_1, \theta_2) = 0\},$$

$$(\eta_1^{\max}, \eta_2^{\max}) = \arg \max_{(\theta_1,\theta_2)} \{\theta_1; \gamma(\theta_1, \theta_2) = 0\},$$

$$(\eta_1^{\min}, \eta_2^{\min}) = \arg \min_{(\theta_1,\theta_2)} \{\theta_1; \gamma(\theta_1, \theta_2) = 0\},$$

$$\beta = \max\{\theta; \gamma(\theta, \theta) = 0\} = \frac{2(r_1 + r_2)}{\sigma_1^2 + \sigma_2^2},$$

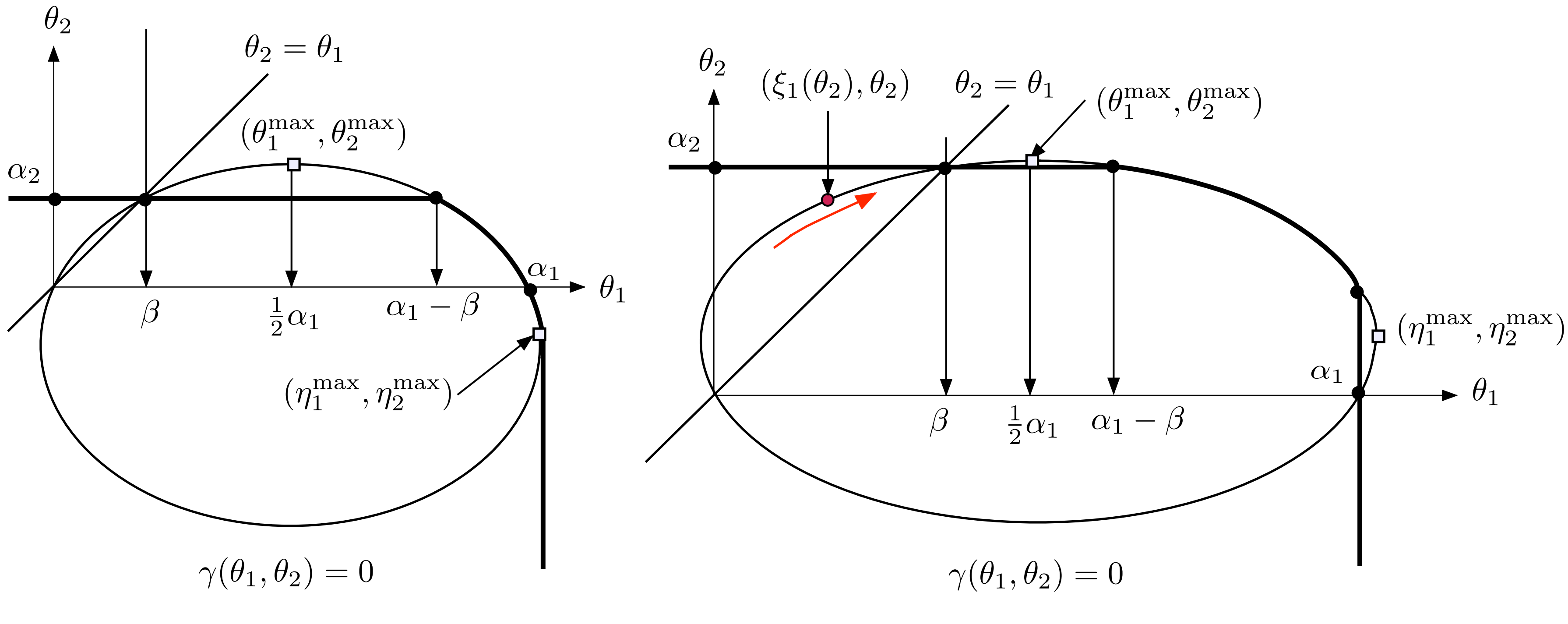
$$\xi_1(\theta_2) = \min\{\theta_1; \gamma(\theta_1, \theta_2) = 1\},$$

$$\xi_2(\theta_1) = \max\{\theta_2; \gamma(\theta_1, \theta_2) = 1\}.$$

From the stability condition $r_1 > 0, r_1 + r_2 > 0$, we have $\beta > 0$.

Shape of the convergence domain 1

Denote the interior of $\mathcal{D} = \{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi(\boldsymbol{\theta}) < \infty\}$ by $\mathcal{D}^{\rm o}$.The following figures are for the case that $\beta < \frac{1}{2}\alpha_1$.

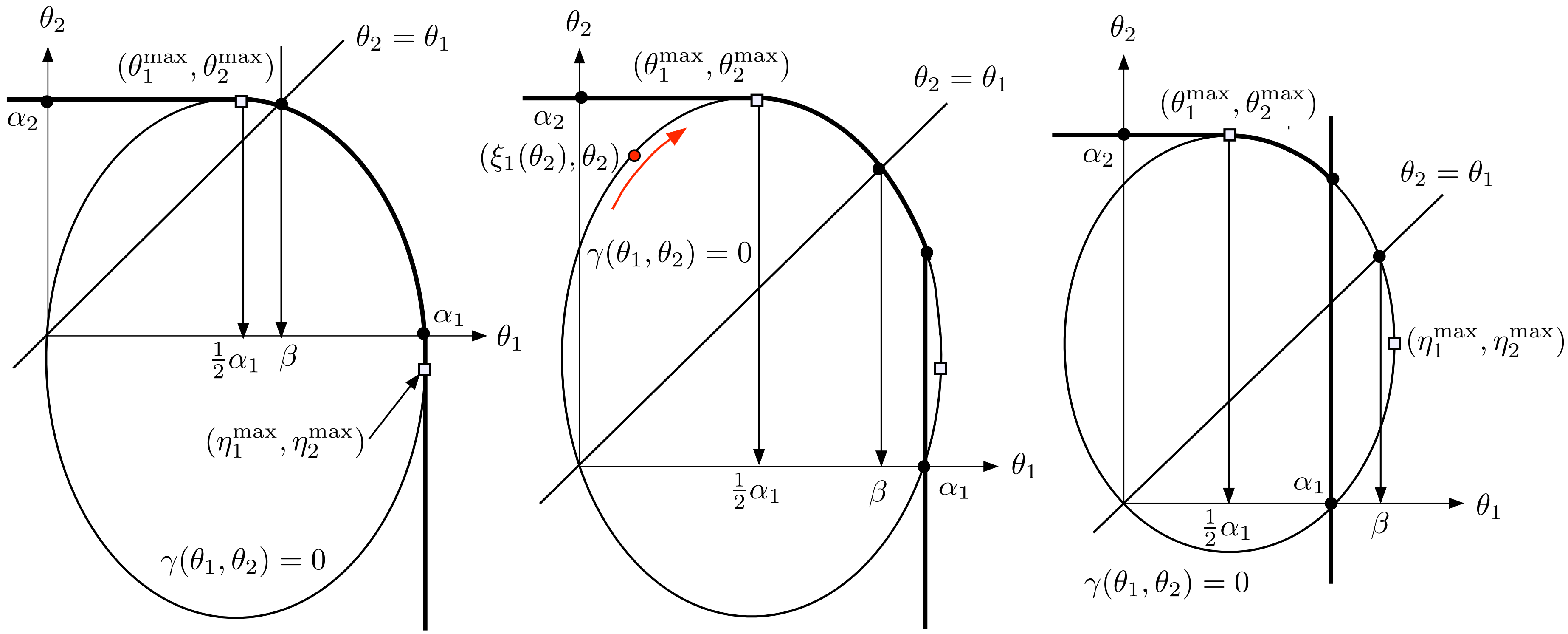


(a−) $\beta < \frac{1}{2}\alpha_1$ with $r_2 < 0$ (a+) $\beta < \frac{1}{2}\alpha_1$ with $r_2 > 0$

The domain $\mathcal{D}^{\rm o}$ is under the bold face curves.

Shape of the convergence domain 2

The following figures are for the case that $\beta > \frac{1}{2}\alpha_1$.



(b−) $\frac{1}{2}\alpha_1 \le \beta \le \alpha_1$ with $r_2 < 0$ (b+) $\frac{1}{2}\alpha_1 \le \beta \le \alpha_1$ with $r_2 > 0$ (c) $\alpha_1 < \beta$

The domain $\mathcal{D}^{\circ}$ is under the bold face curves.

Answer to the convergence domain

Define $\alpha_1^{\max}$ as

$$\alpha_1^{\max} = \begin{cases} \alpha_1, & \eta_2^{\max} \geq 0, \\ \eta_1^{\max}, & \eta_2^{\max} < 0, \end{cases}$$

Define the following two sets:.

$$\mathcal{D}_1 = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1^{\max}, \theta_2 < \beta_2^{\max}\},$$

$$\mathcal{D}_2 = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \theta_1', \theta_2 < \theta_2', \text{ for some } (\theta_1', \theta_2') \text{ such that } \gamma(\theta_1', \theta_2') \geq 0\}.$$

Then,

Proposition 2 If $\{X_i(t)\}$'s are Brownian motions, then the interior of convergence domain $\mathcal{D}^{\mathrm{o}} = \mathcal{D}_1 \cap \mathcal{D}_2$ (**see Figure 1**).

Remark 2 This result is **straightforwardly extended to the Lévy input case** if $\kappa(\boldsymbol{\theta})$ is finite on a sufficiently large set.

Asymptotics for the simple pole case

Let $\psi(\theta) = \int_0^\infty e^{\theta x}\overline{F}(x)\,dx$ for the tail of distribution $\overline{F}(x)$.

(S1) If there are α, p, q satisfying $p < \alpha < q$ and integer $k \geq 1$ such that

(S1a) $\psi(z)$ is analytic for $p \leq \Re z \leq q$ except for $z = \alpha$,

(S1b) $\psi(z)$ uniformly converges to 0 as $z \to \infty$ for $p \leq \Re z \leq q$, and the integral $\frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{-ixy}\psi(q+iy)dy$ uniformly converges for $x > T$,

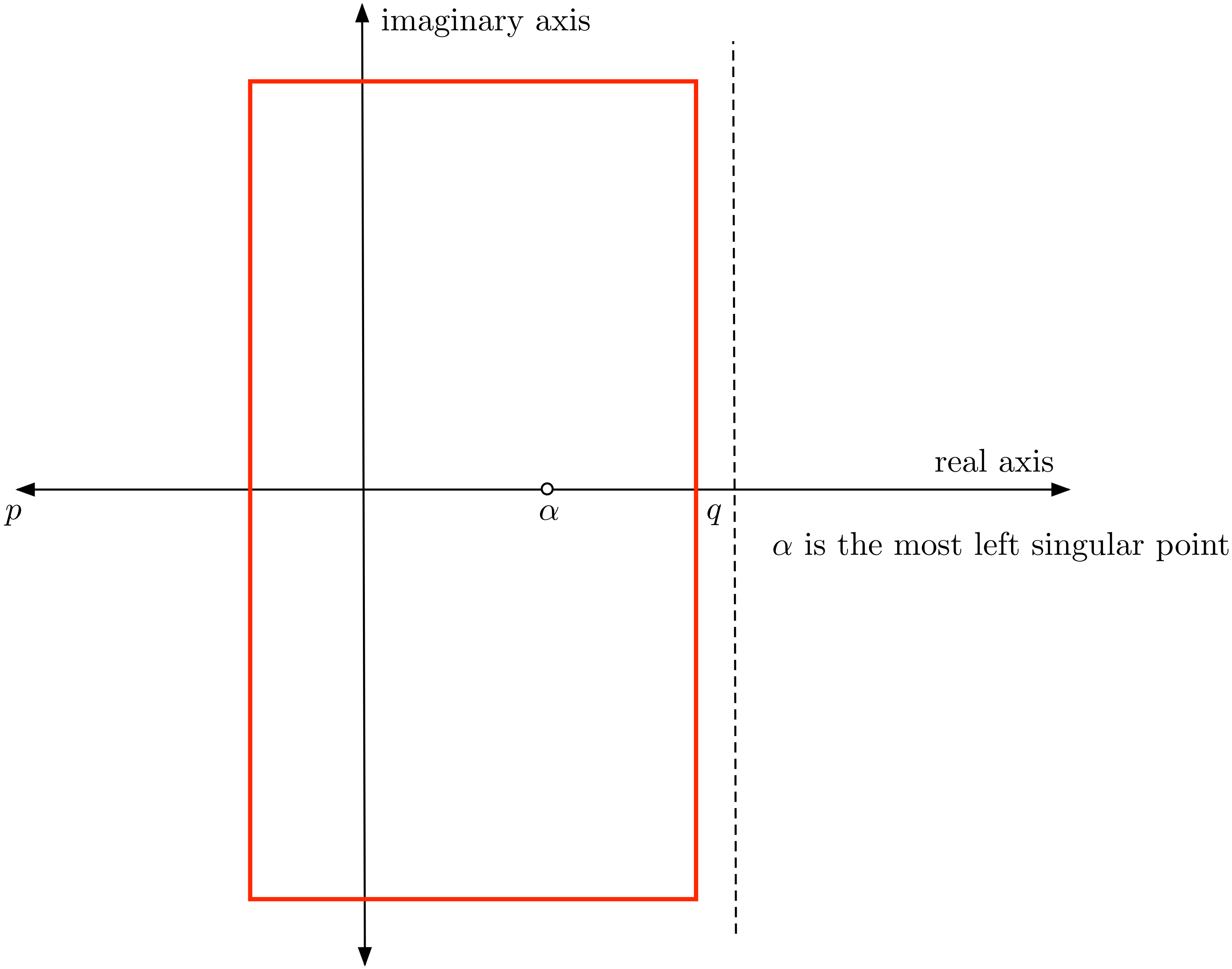
(S1c) $\lim_{z\to} (\alpha - z)^k \psi(z) = C_1'$ for some $C_1' > 0$,

then

$$\overline{F}(x) = \frac{C_1'}{\Gamma(k)} x^{k-1} e^{-x}\ (1 + o(1)),$$

where $\Gamma(z)$ is the gamma function.

Counter integral for the simple pole



Asymptotics for the branch point case

(S2) If there are $\alpha > 0$ and $\delta \in [0, \frac{\pi}{2})$ such that

(S2a) $\psi(z)$ is analytic in the region:

$\mathcal{G}(\delta) \equiv \{z \in \mathbb{C}; \Re z > 0, z \neq \alpha, |\arg(z - \alpha)| > \delta\}$,

where $\arg z$ is the principal part of the argument of complex number z,

(S2b) $\psi(z) \to 0$ as $|z| \to \infty$ for $z \in \mathcal{G}(\delta)$,

(S2c) for some constant K and non integer real number s

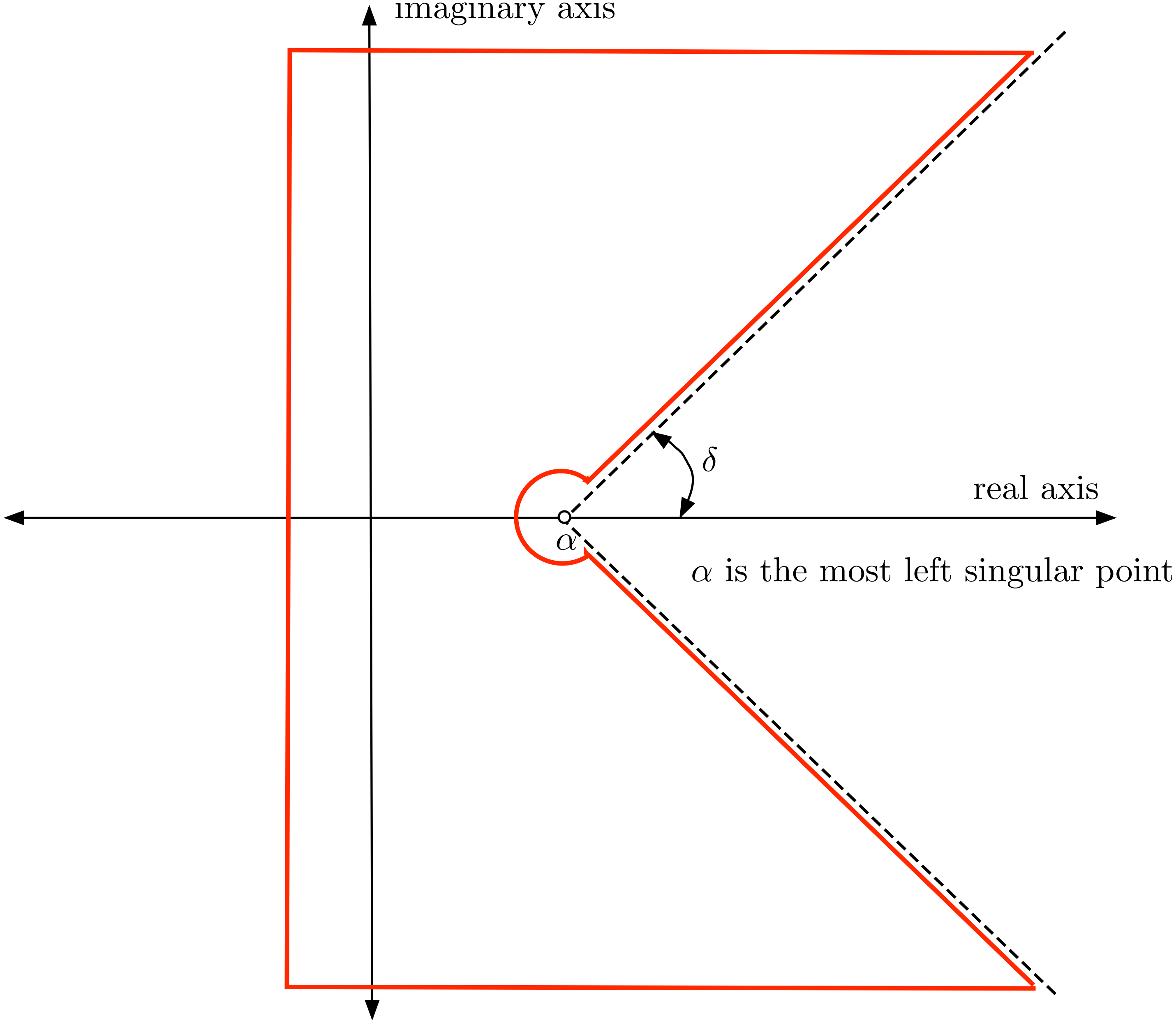
$$\psi(z) = K\,1(s > 0) - C_2'(\alpha - z)^s + o((\alpha - z)^s),$$

for $\mathcal{G}(\delta) \ni z \to \alpha$,

where K must be $\psi(\alpha)$ if $s > 0$, then

$$\overline{F}(x) = \frac{C_2'}{\Gamma(-s)} x^{-s-1} e^{-x}\ (1 + o(1))\ .$$

Counter integral for the branch point



The Brownian input case

Theorem 1 $P(L_2 > x)$ has the following exact asymptotics.

(1a) $\beta < \frac{1}{2}\alpha_1 \Rightarrow h(x) = C_1 e^{- \quad x}$.

(2b) $\frac{1}{2}\alpha_1 = \beta \Rightarrow h(x) = C_2 x^{-\frac{1}{2}} e^{-\theta_2^{\max} x}$.

(3c) $\frac{1}{2}\alpha_1 < \beta \Rightarrow h(x) = C_3 x^{-\frac{3}{2}} e^{-\theta_2^{\max} x}$.

where C_1, C_2, C_3 are positive constants obtained in terms of $\varphi_2(\theta)$, and

$$\alpha_1 = \frac{2r_1}{\sigma_1^2}, \quad \beta = \frac{2(r_1 + r_2)}{\sigma_1^2 + \sigma_2^2}, \quad \theta_2^{\max} = \frac{1}{\sigma_2^2} \quad r_2 + \sqrt{\left(r_2^2 + r_1^2 \frac{\sigma_2^2}{\sigma_1^2}\right) \left(\right.}$$

Remark 3 These three cases corresponds with those of a two-dimensional skip free and reflected random walk on the nonnegative integer quadrant (called a double QBD in [5]), but $h(x) = Cxe^{- \quad 1 x}$ **occurs in the latter case**.

The Lévy input case

Proposition 3 If $\gamma(\theta, \theta) = 0$ has a positive solution, denoted by β, in addition to some finiteness conditions on γ, then, letting $\theta_1^{\max} = \xi_1(\theta_2^{\max})$, we have

(2a) If $\beta < \theta_1^{\max}$, then $\int_0^x e^{\ \ u} P(L_2 > u)du \sim C_1 x$.

(2b) If $\beta = \theta_1^{\max}$, then $\int_0^x e^{\theta_2^{\max} u} P(L_2 > u)du \sim 2C_2 x^{\frac{1}{2}}$.

(2c) If $\beta > \theta_1^{\max}$, then $\int_x^\infty e^{\theta_2^{\max} u} P(L_2 > u)du \sim 2C_3 x^{-\frac{1}{2}}$.

Here, $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$, and

$$C_1 = \frac{\varphi_2(\ \ )}{(\xi_1'(\ \ ) - 1)(\tilde{\kappa}_2(\ \ ) - r_2\ \ )} \text{ and } C_2 = \frac{\theta_2^{\max} \varphi_2(\theta_1^{\max})}{\tilde{\kappa}_2(\theta_2^{\max}) - r_2 \theta_2^{\max}} \sqrt{\frac{-\xi_2''(\theta_1^{\max})}{2\pi}},$$

where $\xi_1'(\beta) = \frac{\tilde{\kappa}_2'(\ \ ) - r_2}{r_1 - \tilde{\kappa}_1'(\ \ )}$ and $\xi_2''(\theta_1^{\max}) = \frac{\tilde{\kappa}_1''(\theta_1^{\max})}{r_2 - \tilde{\kappa}_2'(\theta_2^{\max})}$, and

$$C_3 = \frac{\theta_2^{\max}(\varphi_2(\theta_1^{\max}) + (\theta_2^{\max} - \theta_1^{\max})\varphi_2'(\theta_1^{\max}))}{(\tilde{\kappa}_2(\theta_2^{\max}) - r_2\theta_2^{\max})(\theta_2^{\max} - \theta_1^{\max})^2 \sqrt{-2\pi\xi_2''(\theta_1^{\max})}}.$$

Conjectures on the Lévy input case

Conjecture 1 $P(L_2 > x)$ has the following exact asymptotics.

(3a) $\beta < \frac{1}{2}\alpha_1 \Rightarrow h(x) = C_1 e^{-\ \ x}$.

(3b) $\frac{1}{2}\alpha_1 = \beta \Rightarrow h(x) = C_2 x^{-\frac{1}{2}} e^{-\theta_2^{\max} x}$.

(3c) $\frac{1}{2}\alpha_1 < \beta \Rightarrow h(x) = C_3 x^{-\frac{3}{2}} e^{-\theta_2^{\max} x}$.

where C_1, C_2, C_3 are positive constants.

Remark: If $P(L_2 > x) \sim x^u e^{-vx}$ for some u, v, then Proposition 3 proves this conjecture.

Why is it difficult to prove the conjecture ?

- To apply the complex inversion, we need to find the function $\xi_1(z)$ which is implicitly determined by

$$\gamma(\xi_1(z), z) = 1$$

and analytic in some left-half plane.

Remarks on extensions

The present formulation can be extended to a n-node network with a n-dimesional Lévy input and reflection matrix R such that

$$\boldsymbol{L}(t) = \boldsymbol{L}(0) + \boldsymbol{X}(t) - tR\boldsymbol{c} + R\boldsymbol{Y}(t),$$

where $R = I - P^{\mathrm{T}}$ for routing matrix P, $\boldsymbol{X}(t)$ is the Lévy process, $\boldsymbol{L}(t)$ is a buffer content and $\boldsymbol{Y}(t)$ is a regulator. Under the stability condition $(I - P)^{-1}E(\boldsymbol{X}(1)) < \boldsymbol{c}$, the mgf φ of the stationary distribution of $\boldsymbol{L}(t)$ satisfies

$$\gamma(\boldsymbol{\theta})\varphi(\boldsymbol{\theta}) = \langle \boldsymbol{\theta}, R\underline{\boldsymbol{\varphi}}(\boldsymbol{\theta})\rangle, \tag{7}$$

where $\gamma(\boldsymbol{\theta}) = \langle \boldsymbol{\theta}, R\boldsymbol{c}\rangle - \kappa(\boldsymbol{\theta})$ with Lévy exponent $\kappa(\boldsymbol{\theta})$, and

$$\underline{\boldsymbol{\varphi}}(\boldsymbol{\theta}) = (\varphi_1(\boldsymbol{\theta}_1[0]), \ldots, \varphi_n(\boldsymbol{\theta}_n[0]))^{\mathrm{T}},$$

where $\varphi_i(\boldsymbol{\theta}_i[0]) = E_\pi \int_0^1 e^{\langle \boldsymbol{\theta}_i[0], \boldsymbol{L}(u)\rangle}\, dY_i(u)$.

How to get the domain of φ ?

Let us consider the case that $n = 3$. In this case,

$$\gamma(\boldsymbol{\theta})\varphi(\boldsymbol{\theta}) = \gamma_1(\boldsymbol{\theta})\varphi_1(\theta_2, \theta_3) + \gamma_2(\boldsymbol{\theta})\varphi_2(\theta_1, \theta_3) + \gamma_3(\boldsymbol{\theta})\varphi_3(\theta_1, \theta_2),$$

where

$$\gamma_i(\boldsymbol{\theta}) = \theta_i - (p_{i1}\theta_1 + p_{i2}\theta_2 + p_{i3}\theta_3), \qquad i = 1, 2, 3.$$

Letting $\theta_1 = \theta_2 = 0$ and rearranging the terms,

$$\gamma(\boldsymbol{\theta})\varphi(\theta_1, 0, 0) - \gamma_2(\boldsymbol{\theta})\varphi_2(\theta_1, 0) - \gamma_3(\boldsymbol{\theta})\varphi_3(\theta_1, 0) = \gamma_1(\boldsymbol{\theta})\varphi_1(0, 0),$$

Hence, by similar argumenrs, for $\boldsymbol{\theta} > \mathbf{0}$,

$\gamma(\theta_1, 0, 0) > 0 \Rightarrow \varphi(\theta_1, 0, 0), \varphi_2(\theta_2, 0), \varphi_3(\theta_1, 0) < \infty.$

$\gamma(0, \theta_2, 0) > 0 \Rightarrow \varphi(0, \theta_2, 0), \varphi_1(\theta_2, 0), \varphi_3(0, \theta_3) < \infty.$

$\gamma(0, 0, \theta_3) > 0 \Rightarrow \varphi(0, 0, \theta_3), \varphi_1(0, \theta_3), \varphi_2(0, \theta_3) < \infty.$

These are building blocks to find the convergence domain of $\varphi(\boldsymbol{\theta})$.

Problems

- Exact asymptotics for the 2-node tandem fluid queue driven by the Lévy input.
- Exact asymptotics in all directions for the 2-dimensional Brownian and Lévy networks
- Rough asymptotics for the n-dimensional Brownian and Lévy networks.

One can reasonably conjecture the solutions for these problems. However, their verification would require

- local limit theorems along the boundary faces,
- interference across boundary faces,
- lower bounds for logarithmic tail probabilities.

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